

# The New Triangle in Normed Space

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## The New Triangle in Normed Space

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### Abstract

This paper discusses the concept of a new triangle in normed space. This triangle is the development of triangle in Euclidean space and inner product space. Discussion about triangle in a normed space is defined using the Wilson angle. Next will be attested to some fundamental trait of a triangle in the normed space.

**AMS subject classification:** 11R52, 42C40.

**Keywords:** Norm space, Euclid space, Wilson angle, triangle.

### 1. Introduction

Bottema O, 2008, has discussed the angle between two lines in Euclid space  $\mathbb{R}^2$  by using dot product [2]. Anton H, 2010, with the idea of inequality Cauchy - Schwarz has provided a sense of the angle between two vectors in the inner product space [1]. Furthermore, Gunawan H, Lindiarni J, and Neswan O, 2008, in his writing has also, discussed some angles between the two vectors in the normed space, i.e angle  $P$ , angle  $I$ , angle  $g$  [3, 4, 5, 6]. As well as Milicic PM, 2011, has covered the angle  $Thy$  [4] Valentine and Waymant has covered the Wilson angle [8]. Milicic PM, 2007, also already discuss about the angle  $B$  and angle  $g$ . This paper will define a new triangle in a normed space with using an angle Wilson. The Wilson angle is introduced by Valentine and Waymant [8]. The study of the Wilson angle is discussed as follows.

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Let  $(V, \|\cdot\|)$  be is a normed space, defined non-linear function,

$$2\langle a, b \rangle := \|a\|^2 + \|b\|^2 - \|a - b\|^2, \quad \forall a, b \in V. \quad (1)$$

From the norms it is known that:

$$\begin{aligned} \left| \|a\| - \|b\| \right|^2 &\leq \|a - b\|^2 \\ \Leftrightarrow \|a\|^2 - 2\|a\| \cdot \|b\| + \|b\|^2 &\leq \|a - b\|^2 \\ \Leftrightarrow \langle a, b \rangle &\leq \|a\| \cdot \|b\| \end{aligned} \quad (2)$$

On the other side is obtained:

$$\begin{aligned} \|a - b\|^2 &\leq (\|a\| + \|b\|)^2 \\ \Leftrightarrow \|a - b\|^2 - \|a\|^2 - \|b\|^2 &\leq 2\|a\| \cdot \|b\| \\ \Leftrightarrow -\langle a, b \rangle &\leq \|a\| \cdot \|b\| \end{aligned} \quad (3)$$

From the equation (2) and (3) are obtained :

$$|\langle a, b \rangle| \leq \|a\| \cdot \|b\|, \quad \forall a, b \in V. \quad (4)$$

which fulfills the inequality Cauchy-Schwarz [1]. From the equation (4) defined angle Wilson as an angle between two vectors  $a$  and  $b$  has properties:

$$\angle(a, b) := \arccos\left(\frac{\|a\|^2 + \|b\|^2 - \|a - b\|^2}{2\|a\| \cdot \|b\|}\right) \quad (5)$$

Let  $c = a - b$  be, then of the equation (5) obtained by cosine rules [9]:

$$\|c\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\| \cdot \|b\| \cos \angle(a, b). \quad (6)$$

From the equation (6) can be obtained the following sine rules:

$$\|a\| \cdot \|b\| \sin \angle(a, b) = K \quad (7)$$

with  $K = 2\sqrt{(s - \|a\|)(s - \|b\|)(s - \|c\|)}$ , and  $2s = \|a\| + \|b\| + \|c\|$

## 2. Result

**Definition 2.1.** Let  $(V, \|\cdot\|)$  be, is a normed space, for  $a, b, c \in V \setminus \{0\}$ , defined a triangle that is symbolized  $\Delta[a, b, c]$  is  $\{a, b, c\}$  have the properties  $a + c = b$  completed with Wilson angles  $\angle(a, b)$ ,  $\angle(b, c)$  and  $\angle(-a, c)$ .

**Theorem 2.2.** Let  $(V, \|\cdot\|)$  be, is a normed space, In the triangle  $\Delta[a, b, c]$  that, cosine rules  $\Leftrightarrow$  sine rules.

Proof. ( $\Rightarrow$ ) Let  $\cos \angle(a, b) = \left( \frac{\|a\|^2 + \|b\|^2 - \|a - b\|^2}{2\|a\| \cdot \|b\|} \right)$  be, then :

$$\begin{aligned} \sin^2 \angle(a, b) &= 1 - \cos^2 \angle(a, b) \\ &= 1 - \left( \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|} \right)^2 \\ &= \frac{(2\|a\|\|b\|)^2 - (\|a\|^2 + \|b\|^2 - \|c\|^2)^2}{4\|a\|^2\|b\|^2} \\ &= \frac{(2\|a\|\|b\|) - (\|a\|^2 + \|b\|^2 - \|c\|^2)}{((2\|a\|\|b\|) + (\|a\|^2 + \|b\|^2 - \|c\|^2))} \\ &= \frac{\|c\|^2 - (\|a\| - \|b\|)^2(\|a\| + \|b\|)^2 - \|c\|^2}{4\|a\|^2\|b\|^2} \\ &= \frac{(\|a\| + \|b\| + \|c\|)(\|b\| + \|c\| - \|a\|)}{(\|a\| + \|c\| - \|b\|)(\|a\| + \|b\| - \|c\|)} \\ &= \frac{4\|a\|^2\|b\|^2}{(2s)2(s - \|a\|)2(s - \|b\|)2(s - \|c\|)} \\ &= \frac{16(s)(s - \|a\|)(s - \|b\|)(s - \|c\|)}{4\|a\|^2\|b\|^2} \end{aligned}$$

$$\|a\|\|b\| \sin \angle(a, b) = 2\sqrt{(s - \|a\|)(s - \|b\|)(s - \|c\|)} = K.$$

( $\Leftarrow$ ) By eliminating the sine rule it is obtained:

$$\begin{aligned} K^2(\|a\|^2 + \|b\|^2 - \|c\|^2) &= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b, c) + \sin^2 \angle(-a, c) \\ &\quad - \sin^2 \angle(a, b)) \\ &= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b, c) + \sin^2 \angle(-a, c) \\ &\quad - \sin^2 (\angle(b, c) + \angle(-a, c))) \\ &= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b, c) + \sin^2 \angle(-a, c) \\ &\quad - (\sin \angle(b, c) \cos \angle(-a, c) \\ &\quad + \cos \angle(b, c) \sin \angle(-a, c))^2) \\ &= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b, c) + \sin^2 \angle(-a, c) \\ &\quad - \sin^2 \angle(b, c) \cos^2 \angle(-a, c) \\ &\quad - \cos^2 \angle(b, c) \sin^2 \angle(-a, c) \\ &\quad - 2 \sin \angle(b, c) \cos \angle(-a, c) \cdot \cos \angle(b, c) \sin \angle(-a, c)) \end{aligned}$$

$$\begin{aligned}
&= \|a\|^2 \|b\|^2 \|c\|^2 (\sin^2 \angle(b, c)(1 - \cos^2 \angle(-a, c)) + \sin^2 \angle(b, c)(1 - \cos^2 \angle(b, c)) \\
&\quad - 2 \sin \angle(b, c) \cos \angle(-a, c) \cdot \cos \angle(b, c) \sin \angle(-a, c)) \\
&= \|a\|^2 \|b\|^2 \|c\|^2 (\sin^2 \angle(b, c)(\sin^2 \angle(-a, c)) + \sin^2 \angle(b, c)(\sin^2 \angle(b, c)) \\
&\quad - 2 \sin \angle(b, c) \cos \angle(-a, c) \cdot \cos \angle(b, c) \sin \angle(-a, c)) \\
&= 2\|a\|^2 \|b\|^2 \|c\|^2 (\sin^2 \angle(b, c)(\sin^2 \angle(-a, c)) \\
&\quad - \sin \angle(b, c) \cos \angle(-a, c) \cdot \cos \angle(b, c) \sin \angle(-a, c)) \\
&= 2\|a\|^2 \|b\|^2 \|c\|^2 (\sin \angle(b, c) \sin \angle(-a, c)(\sin \angle(b, c) \sin \angle(-a, c)) \\
&\quad - \cos \angle(-a, c) \cdot \cos \angle(-a, c)) \\
&= 2\|a\|^2 \|b\|^2 \|c\|^2 \sin \angle(b, c) \sin \angle(-a, c) \cos \angle(a, b) \\
&= 2\|a\| \|b\| \|c\| \cos \angle(a, b)
\end{aligned}$$

so, the following cosine rules are obtained:

$$\|c\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\| \|b\| \cos \angle(a, b). \quad (8)$$

■

**Theorem 2.3.** Let  $(V, \|\cdot\|)$  be, is a normed space, In the triangle  $\Delta[a, b, c]$  that, cosine rules  $\Leftrightarrow$  side triangle rules.

*Proof.* ( $\Rightarrow$ )

$$\|a\| \cos \angle(a, b) = \|a\| \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\| \cdot \|b\|} \quad (9)$$

$$\|c\| \cos \angle(b, c) = \|c\| \frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\| \cdot \|c\|} \quad (10)$$

by eliminating the equation (9) and (10), then obtained the rules of the triangle side:

$$\|a\| \cos \angle(a, b) + \|c\| \cos \angle(b, c) = \|b\|. \quad (11)$$

( $\Leftarrow$ ) From the equation (11) obtained by triangle side rules:

$$\|a\|^2 = \|a\| \|b\| \cos \angle(a, b) + \|a\| \|c\| \cos \angle(-a, c) \quad (12)$$

$$\|b\|^2 = \|a\| \|b\| \cos \angle(a, b) + \|b\| \|c\| \cos \angle(b, c) \quad (13)$$

$$\|c\|^2 = \|b\| \|c\| \cos \angle(b, c) + \|a\| \|c\| \cos \angle(-a, c) \quad (14)$$

by eliminating the equation (12), (13) and (14) obtained by cosine rules:

$$\|a\|^2 + \|b\|^2 - 2\|a\| \|b\| \cos \angle(a, b) = \|c\|^2. \quad (15)$$

and this puts an end to the proof.  $\blacksquare$

**Theorem 2.4.** Let  $(V, \|\cdot\|)$  be a normed space. In the triangle  $\Delta[a, b, c]$ , then  $\angle(a, b) + \angle(b, c) + \angle(-a, c) = \pi$ .

*Proof.* To prove the theorem, it will first show that:

$$\cos \angle(a, b) \cos \angle(b, c) - \sin \angle(a, b) \sin \angle(b, c) + \cos \angle(-a, c) = 0 \quad (16)$$

By using the rules of cosine (6) and sine rules (7), then equation (16) proved as follows:

$$\begin{aligned} & \cos \angle(a, b) \cos \angle(b, c) - \sin \angle(a, b) \sin \angle(b, c) + \cos \angle(-a, c) \\ &= \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\| \cdot \|b\|} \cdot \frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\| \cdot \|c\|} \\ & \quad + \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\| \cdot \|c\|} - \frac{K}{\|a\| \cdot \|b\|} \cdot \frac{K}{\|b\| \cdot \|c\|} \\ &= \frac{(\|a\|^2 + \|c\|^2 - \|b\|^2)(\|b\|^2 + \|c\|^2 - \|a\|^2)}{4\|a\| \cdot \|b\|^2 \cdot \|c\|} \\ & \quad + \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\| \cdot \|c\|} - \frac{K^2}{\|a\| \cdot \|b\|^2 \cdot \|c\|} \\ &= \frac{(-\|a\|^4 + \|b\|^4 - \|c\|^4)}{4\|a\| \cdot \|b\|^2 \cdot \|c\|} + \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\| \cdot \|c\|} - \frac{K^2}{\|a\| \cdot \|b\|^2 \cdot \|c\|} \\ &= \frac{(-\|a\|^4 + \|b\|^4 - \|c\|^4 + 2\|a\|^2 \cdot \|b\|^2 + 2\|b\|^2 \cdot \|c\|^2) - 4K^2}{4\|a\| \cdot \|b\|^2 \cdot \|c\|} \\ &= \frac{4K^2 - 4K^2}{4\|a\| \cdot \|b\|^2 \cdot \|c\|} \\ &= 0. \end{aligned}$$

then

$$\begin{aligned} & \cos \angle(a, b) \cos \angle(b, c) - \sin \angle(a, b) \sin \angle(b, c) + \cos \angle(-a, c) = 0 \\ \Leftrightarrow & \cos \angle(a, b) \cos \angle(b, c) - \sin \angle(a, b) \sin \angle(b, c) = -\cos \angle(-a, c) \\ \Leftrightarrow & \cos(\angle(a, b) + \angle(b, c)) = -\cos \angle(-a, c) \\ \Leftrightarrow & \cos(\angle(a, b) + \angle(b, c)) = \cos(\pi - \angle(-a, c)) \\ \Leftrightarrow & \angle(a, b) + \angle(b, c) = \pi - \angle(-a, c) \\ \Leftrightarrow & \angle(a, b) + \angle(b, c) + \angle(-a, c) = \pi \end{aligned}$$

and this puts an end to the proof.  $\blacksquare$

**Example 2.5.** Let  $(L^3([0, 1]), \|\cdot\|)$  be a normed space, and Let  $\Delta[a, b, c]$  be,  $\{a, b, c\} \subseteq L^3([0, 1])$  and satisfy  $a + c = b$  with  $a(t) := t^3$ ,  $b(t) := t^2$  and  $c(t) := t^2 - t^3$ , further obtained:

$$\begin{aligned}\|a\| &= \left( \int_0^1 |t^3|^3 dt \right)^{\frac{1}{3}} \\ &= \left( \int_0^1 t^9 dt \right)^{\frac{1}{3}} \\ &= 0,464\end{aligned}$$

$$\begin{aligned}\|b\| &= \left( \int_0^1 |t^2|^3 dt \right)^{\frac{1}{3}} \\ &= \left( \int_0^1 t^6 dt \right)^{\frac{1}{3}} \\ &= 0,523\end{aligned}$$

$$\begin{aligned}\|c\| &= \left( \int_0^1 |t^2 - t^3|^3 dt \right)^{\frac{1}{3}} \\ &= \left( \int_0^1 (t^6 - 3t^7 + 3t^8 - t^9) dt \right)^{\frac{1}{3}} \\ &= 0,106\end{aligned}$$

So, obtained:

$$\begin{aligned}\angle(a, b) &= \arccos \left( \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|} \right) = 10,29 \\ \angle(-a, c) &= \arccos \left( \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|} \right) = 118,27 \\ \angle(b, c) &= \arccos \left( \frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\|\|c\|} \right) = 51,44\end{aligned}$$

then obtained  $\angle(a, b) + \angle(a, b) + \angle(a, b) = \pi$ .

**Theorem 2.6.** Let  $(V, \|\cdot\|)$  be, is a normed space. In the triangle  $\Delta[a, b, c]$  with  $\|a\| = \|c\| \Leftrightarrow \angle(a, b) = \angle(b, c)$ .

*Proof.* ( $\Rightarrow$ ) from triangle side rules:

$$\begin{aligned}\|a\| &= \|b\| \cos \angle(a, b) + \|c\| \cos \angle(-a, c) \\ \|c\| &= \|b\| \cos \angle(b, c) + \|a\| \cos \angle(-a, c)\end{aligned}$$

$\|a\| = \|b\|$ , so obtained  $\cos \angle(a, b) = \cos \angle(b, c)$  and because  $\angle(a, b), \angle(b, c) \in [0, \pi]$  then  $\angle(a, b) = \angle(b, c)$   
 (  $\Leftarrow$  ) Because  $\angle(a, b) = \angle(b, c)$  then from sine rules obtained  $\|a\| \sin \angle(a, b) = \|c\| \sin \angle(b, c)$  so obtained  $\|a\| = \|c\|$ . ■

**Corollary 2.7.** Let  $(V, \|\cdot\|)$  be, normed space. In the triangle  $\Delta[a, b, c]$  with  $\|a\| = \|b\| = \|c\| \Leftrightarrow \angle(a, b) = \angle(b, c) = \angle(-a, c)$ .

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# The New Triangle in Normed Space

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